Divisors on Curves (Har II 6) $\longrightarrow$ Read I6 for basics of curves

We won't prove anything in this section since it is mostly the same as in the classical setting, but your homework is to read this section of Hartshorne carefully.

Def: Let k be algebraically closed. A <u>curve</u> over k is an integral separated scheme X of finite type over k of dimension one. If all the local rings $\mathcal{O}_{X,x}$ are regular, then X is <u>nonsingular</u>. If X is proper $/k$ it's <u>complete</u>. (Recall projective $\Rightarrow$ complete. For nonsingular curves, $\Leftarrow$ holds)

If $f: X \to Y$ is a morphism of curves w/ X complete, then $f(X)$ must be closed and irreducible.

So $f(X)$ is either Y or a point.

$$\begin{tikzcd} X \arrow[r, "\text{must be proper}"] \arrow[dr, "\text{proper}"'] & Y \arrow[d, "\text{separated}"] \\ & \operatorname{Spec} k \end{tikzcd}$$

If $f(X) = Y$, Y must also be complete and since it's dominant (i.e. the image contains the generic point), we get an induced inclusion

$$K(Y) \subseteq K(X),$$

which is a finite algebraic extension. In this case, $f: X \to Y$ is finite, and we define the <u>degree</u> of f to be the degree of the field extension.

A prime divisor on a curve is just a closed point, so an arbitrary divisor is of the form

$$D = \sum n_i \underset{\text{closed points}}{P_i}.$$

Define $\deg D := \sum n_i$.

If $f: X \to Y$ is a finite morphism of nonsingular curves, define $f^*: \operatorname{Div} Y \to \operatorname{Div} X$ as follows:

For $Q \in Y$ a closed point, let $t \in \mathcal{O}_Q$ be a unif. parameter. i.e. $v_Q(t) = 1$, where $v_Q: K(Y) \to \mathbb{Z}$ is the valuation corr. to Q. Set

$$f^* Q := \sum_{P \mapsto Q} v_P(t) \cdot P,$$

which is a finite sum since f is finite. $f^*(g) = (g)$ for $g \in K(Y)$ (check!), so f^* preserves linear equivalence, and thus induces $f^*: \operatorname{Cl} Y \to \operatorname{Cl} X$.

Can show: if $Q \in Y$ a closed point, then $\deg f^* Q = \deg f$. Thus, for any divisor $D \in \operatorname{Cl} Y$, $\deg f^* D = \deg D \cdot \deg f$.

We showed that principal divisors on $\mathbb{P}^1$ all have degree 0. This is true for any nonsingular curve:

Claim: If X is a complete nonsingular curve, $f \in K(X)^*$, then $\deg(f) = 0$.

Pf: If $f \in k$, then f is a unit in every DVR in $K(X)$, so $(f) = 0$, so assume $f \notin k$. Then since k is algebraically closed, $k(f) \cong K(\mathbb{P}^1) \subseteq K(X)$, which determines a morphism $\varphi : X \to \mathbb{P}^1$. ✱

Note that in $k(x) = K(\mathbb{P}^1)$,

$(x) = \{0\} - \{\infty\}$

valuation in $k[x]_{(x)}$ is one

valuation in $k[1/x]_{(1/x)}$ is -1.

Thus, $(f) = \varphi^*(\underbrace{\{0\} - \{\infty\}}_{\deg 0})$, so $\deg(f) = 0$.

✱ Recall that there's a 1-1 corr:

$$\left\{\begin{matrix}\text{dominant} \\ \text{maps betwn} \\ \text{nonsingular} \\ \text{projective} \\ \text{curves} \\ C' \to C\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{homo-} \\ \text{morphisms} \\ K(C) \to K(C')\end{matrix}\right\}$$

Thus, we have a degree homomorphism on the class group of X:

$$\deg : \mathrm{Cl}\, X \twoheadrightarrow \mathbb{Z},$$

which is well-defined and surjective, since e.g. $P \mapsto 1$.

However, this map is an isomorphism $\iff X \cong \mathbb{P}^1$:

($\Leftarrow$) we've already shown for any $\mathbb{P}^n$.

($\Rightarrow$): If $P \neq Q$ in X and $P \sim Q$, then $P - Q = (f)$ for some $f \in K(X)^*$, which then induces $\varphi : X \to \mathbb{P}^1$ as above. So

$$\varphi^*(\{0\} - \{\infty\}) = P - Q, \text{ so } \varphi^*(\{0\}) = P, \text{ so } \varphi \text{ must have}$$

degree 1. Thus, $K(X) = K(\mathbb{P}^1)$, so φ is an isomorphism.

For more details about divisors on curves, see Har IV.